Tracks, Lie's, and Exceptional Magic

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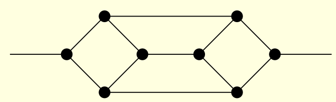
Q : What is the group theoretic weight for QCD diagram



$= ?$

A :

1. new notation: invariant tensors $\leftrightarrow$ "Feynman" diagrams
2. new computational method: diagrammatic, start $\rightarrow$ finish
3. new relations: "negative dimensions" $SO(n) \leftrightarrow Sp(-n)$, $E_7 \leftrightarrow SO(4)$, etc.
4. new classification: primitive invariants $\rightarrow$ all semi-simple Lie algebras

Magic Triangle

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						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Part I: Lie groups, a review

1. linear transformations
2. invariance groups
3. birtrack notation
4. primitive invariants
5. reduction of multi-particle states
6. Lie algebras

Linear transformations

defining rep of group $\mathcal{G}$:

$$G : V \to V\,, \qquad [n \times n] \text{ matrices } G_a{}^b \in \mathcal{G}$$

defining multiplet: particle wave function $q \in V$ transforms as $V \to V$

$$q'_a = G_a{}^b q_b\,, \qquad a, b = 1, 2, \ldots, n$$

conjugate multiplet: "antiparticle" wave function $\bar{q} \in \bar{V}$ transforms as $\bar{V} \to \bar{V}$

$$q'^a = G^a{}_b q^b$$

tensors: multi-particle states transform as $V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$

$$p'_a q'_b r'^c = G_a{}^f G_b{}^e G^c{}_d p_f q_e r^d$$

Note: repeated indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b\,,$$

unless explicitly stated otherwise.

Invariants

A multinomial

$$H(\overline{q}, \overline{r}, \dots, s) = h_{ab\dots}{}^{\dots c}\, q^a r^b \dots s_c$$

is an invariant of the group $\mathcal{G}$ if for all $G \in \mathcal{G}$ and any set of vectors $q, r, s, \dots$ it satisfies

invariance condition: $$H(\overline{Gq}, \overline{Gr}, \dots Gs) = H(\overline{q}, \overline{r}, \dots, s) \,.$$

Invariance group

Definition. An invariance group $\mathcal{G}$ is the set of all linear transformations which leave invariant

$$p_1(x,\bar{y})=p_1(Gx,\bar{y}G^\dagger)\,, \qquad p_2(x,y,z,\ldots)\;=\;p_2(Gx,Gy,Gz\ldots)\,, \qquad \ldots\,.$$

a finite list of primitive invariants:

$$\mathbf{P}=\{p_1,p_2,\ldots,p_k\}$$

No other primitive invariants exist.

(a more precise statement in what follows)

Notation

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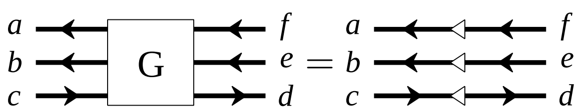
tensorial index notation:

$$p'_a q'_b r'^c = G_{ab}{}^c{}_{,d}{}^{ef} p_f q_e r^d \,, \qquad G_{ab}{}^c{}_{,d}{}^{ef} = G_a{}^f G_b{}^e G^c{}_d$$

collective indices notation:

$$q'_\alpha = G_\alpha{}^\beta q_\beta \qquad {}_\alpha = \left\{ \begin{array}{c} \quad c \\ ab \quad \end{array} \right\} \,, \qquad {}^\beta = \left\{ \begin{array}{c} \quad ef \\ d \quad \end{array} \right\}$$

diagrammatic notation:

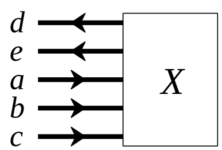


Birdtracks

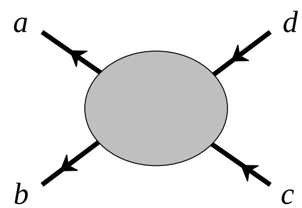
agglomerations of invariant tensors → birdtracks (group-theoretical "Feynman" diagrams)

Invariant tensors → vertices (blobs with external legs)

$$X_\alpha = X^{abc}_{de} =$$

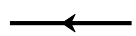


$$, \qquad h^{cd}_{ab} =$$



Contractions → propagators (Kronecker deltas)

$$\delta^a_b = b$$



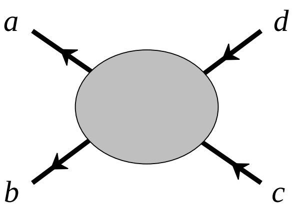
$$a$$

Birdtracks rule

Rules:

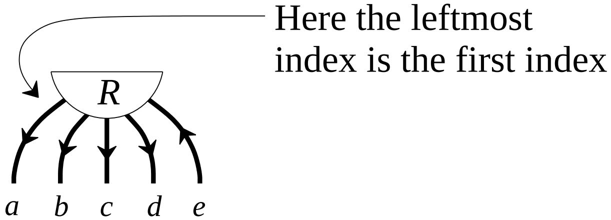
(1) Direct arrows from upper indices "downward" toward the lower indices:

$$h^{cd}_{ab} =$$



(2) Indicate which in (out) arrow corresponds to the first upper (lower) index:

$$R^{e}_{abcd} =$$



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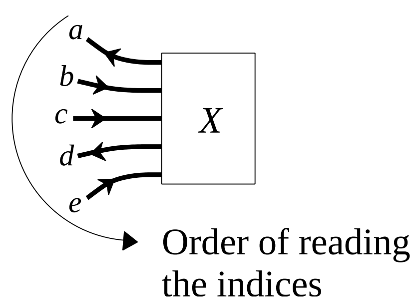
(3) Read in the counterclockwise order around the vertex:

$$x^{bce}_{ad} =$$



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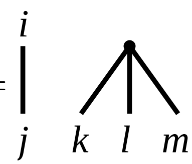
Composed invariants, tree invariants

Definition. A composed invariant tensor is a product and/or contraction of invariant tensors.

Examples:

$$\delta_{ij}\epsilon_{klm} =$$ 

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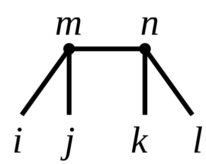
$$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$$ 

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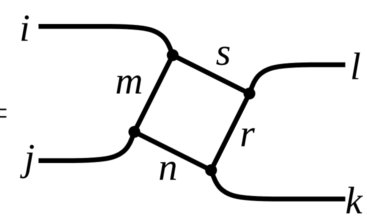
Corresponding invariants:

product $L(x,y)V(z,r,s)$; index contraction $V(x,y,\frac{d}{dz})\,V(z,r,s)$.

Definition. A tree invariant involves no loops of index contractions.

Example: The above tensors are tree invariants. The tensor

$$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} =$$ 

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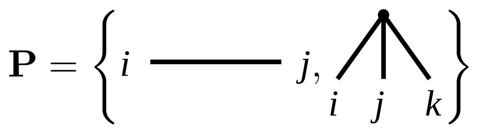
with interal loop indices m, n, r, s summed over, is not a tree invariant.

Primitive invariants

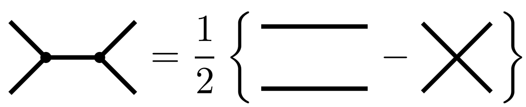
Definition. An invariant tensor is primitive if it cannot be expressed as a combination of tree invariants composed of other primitive invariant tensors.

Example:

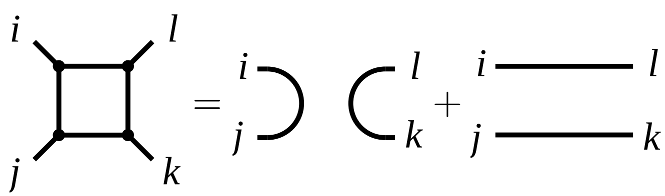
Kronecker delta and Levi-Civita tensor are the primitive invariant tensors of our 3-dimensional space.



4-vertex loop is not a primitive, because the Levi-Civita relation



reduces it to a sum of tree contractions:



Primitiveness assumption

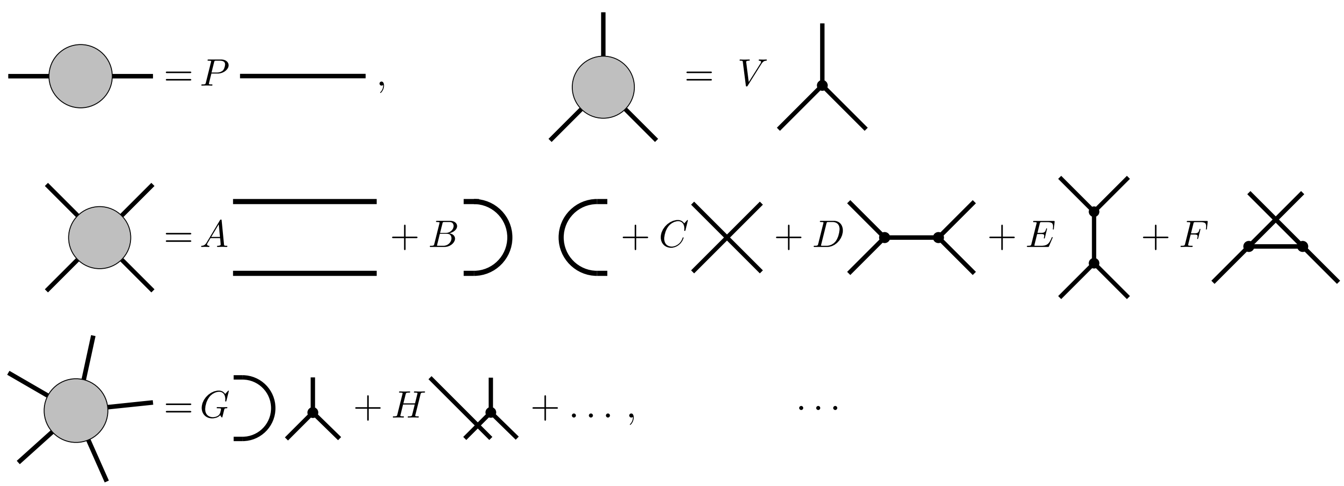
Let $T = \{\mathbf{t}_0, \mathbf{t}_1 \ldots \mathbf{t}_r\}$ = a maximal set of r linearly independent tree invariants $\mathbf{t}_\alpha \in V^p \otimes \bar{V}^q$.

Primitiveness assumption. Any invariant tensor $h \in V^p \otimes \bar{V}^q$ can be expressed as a linear sum over the basis set T.

$$h = \sum_T h^\alpha \mathbf{t}_\alpha \, .$$

Example:

Given primitives $P = \{\delta_{ij}, f_{ijk}\}$, any invariant tensor $h \in V^p$ (here denoted by a blob) is expressible as



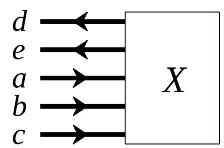
Hermitian conjugation

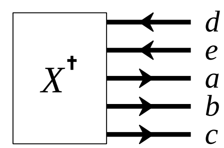
Hermitian conjugation

(a) exchanges the upper and the lower indices, *ie.* reverses arrows

(b) it reverses the order of the indices, *ie.* transposes a diagram into its mirror image.

Example: A tensor and its conjugate:

$$X_\alpha = X_{de}^{abc} =$$ 

$$, \qquad X^\alpha = X_{cba}^{ed} =$$ 

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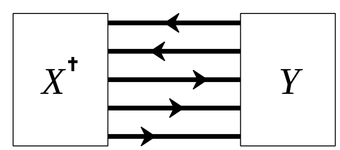
Motivation: contraction $X^\dagger X = |X|^2$ can be drawn in a plane.

Example: contraction of tensors $X^\dagger$ and Y:

$$X^\alpha Y_\alpha = X_{a_q \dots a_2 a_1}^{b_p \dots b_1} Y_{b_1 \dots b_p}^{a_1 a_2 \dots a_q} =$$ 

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Real defining space, $V = \bar{V}$: no distinction between up and down indices, lines carry no arrows

$$\delta_i^j = \delta_{ij} = i \;\text{———}\; j$$

Hermitian matrices

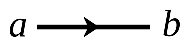
Invariant tensor $M \in V^{p+q} \otimes \bar{V}^{p+q}$ is a hermitian matrix

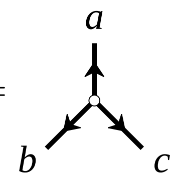
$$M : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$$

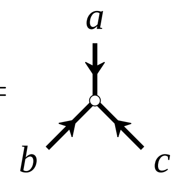
if it is invariant under transposition and arrow reversal.

Example:

Given the 3 primitive invariant tensors:

$\delta_a^b =$ 

, $d_{abc} =$ 

, $d^{abc} = (d_{abc})^* =$ 

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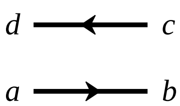
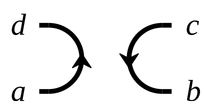
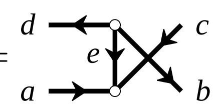
(d_{abc} fully symmetric) can construct 3 hermitian matrices $M : V \otimes \bar{V} \to V \otimes \bar{V}$

$\delta_b^a \delta_d^c =$ 

, $\delta_d^a \delta_b^c =$ 

, $d^{ace} d_{ebd} =$ 

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Self-dual under transposition and arrow reversal.

Projection operators

Hermitian matrix M is diagonalizable by a unitary transformation C

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \dots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

Removing a factor $(M - \lambda_j \mathbf{1})$ from its characteristic equation $\prod (M - \lambda_i \mathbf{1}) = 0$ yields a

projection operator:

$$P_i = \prod_{j \neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} = C^\dagger \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & 0 & \\ & & 0 & & & & \\ & & & \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & 1 \end{pmatrix} & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} C$$

for each distinct eigenvalue of M.

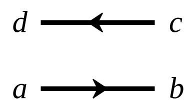
$U(n)$ invariant matrices

Example

$U(n)$ is the invariance group of the norm of a complex vector $|x|^2 = \delta^a_b x^b x_a$.

only one primitive invariant tensor: $\delta^a_b = a \longrightarrow b$

Can construct 2 invariant hermitian matrices $M \in V^2 \otimes \bar{V}^2$:

identity : $\mathbf{1}^{a\,c}_{d,b} = \delta^a_b \delta^c_d =$ 

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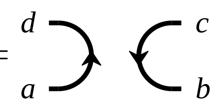
trace : $T^{a\,c}_{d,b} = \delta^a_d \delta^c_b =$ 

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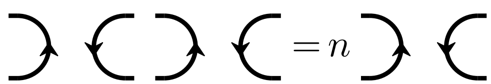
The characteristic equation for T in tensor, birdtrack, matrix notation:

$$T^{a\,f}_{d,e} T^{e\,c}_{f,b} = \delta^a_d \delta^f_e \delta^e_f \delta^c_b = n\, T^{a\,c}_{d,b}\,,$$



$$T^2 = nT\,.$$

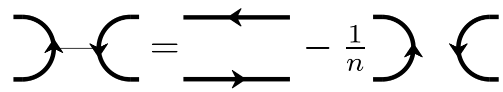
$\delta^e_e = n =$ the dimension of the defining vector space V.

$U(n)$ reduction

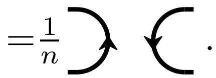
The roots of the characteristic equation $T^2 = nT$ are $\lambda_1 = 0$, $\lambda_2 = n$.

The corresponding projection operators decompose $U(n) \to SU(n) \oplus U(1)$:

$SU(n)$ adjoint rep: $P_1 = \frac{T - n\mathbf{1}}{0 - n} = \mathbf{1} - \frac{1}{n}T$



$U(n)$ singlet: $P_2 = \frac{T - 0\cdot\mathbf{1}}{n-1} = \frac{1}{n}T$



Infinitesimal transformations

Infinitesimal unitary transformation, its action on the conjugate space:

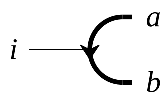
$$G_a^b = \delta_a^b + i\epsilon_j (T_j)_a^b \,, \qquad (G^\dagger)_b^a = \delta_b^a - i\epsilon_j (T_j)_b^a \,, \qquad |D_a^b| \ll 1 \,.$$

is parametrized by

$$N = \text{dimension of the group (Lie algebra, adjoint rep)} \leq n^2$$

real parameters ϵ_j. The adjoint representation matrices $\{T_1, T_2, \cdots, T_N\}$ are generators of infinitesimal transformations, drawn as

$$\frac{1}{\sqrt{a}}(T_i)_b^a = $$



$$a, b = 1, 2, \ldots, n \,, \quad i = 1, 2, \ldots, N \,,$$

where a is an (arbitrary) overall normalization.

The adjoint representation Kronecker delta will be drawn as a thin straight line

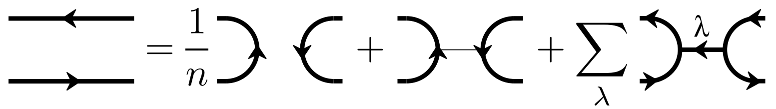
$$\delta_{ij} = i \;\text{———}\; j \,, \qquad i, j = 1, 2, \ldots, N \,.$$

Adjoint representation

Consider the decomposition of $V \otimes \bar{V}$ into (ir)reducible subspaces; the adjoint subspace is always contained in $V \otimes \bar{V}$:

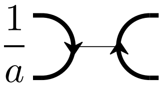
$$\mathbf{1} = \frac{1}{n} T + P_A + \sum_{\lambda \neq A} P_\lambda$$

$$\delta^a_d \delta^c_b = \frac{1}{n} \delta^a_b \delta^c_d + (P_A)^a_b{}_,{}^c_d + \sum_{\lambda \neq A} (P_\lambda)^a_b{}_,{}^c_d$$



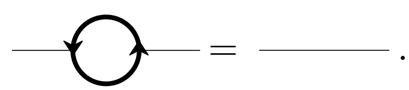
where the adjoint rep projection operators is drawn in terms of the generators:

$$(P_A)^a_b{}_,{}^c_d = \frac{1}{a} (T_i)^a_b (T_i)^c_d = \frac{1}{a}$$



The arbitrary normalization a cancels out in the projection operator orthogonality condition

$$\mathrm{tr}\,(T_i T_j) = a\,\delta_{ij}$$



Invariance, infinitesimally

Invariant tensor h is unchanged under an infinitesimal transformation $G : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$:

$$G_\alpha{}^\beta h_\beta = (\delta_\alpha{}^\beta + \epsilon_j (T_j)_\alpha{}^\beta) h_\beta + O(\epsilon^2) = h_\alpha \, ,$$

so generators of infinitesimal transformations annihilate invariant tensors

$$T_i h = 0 \, .$$

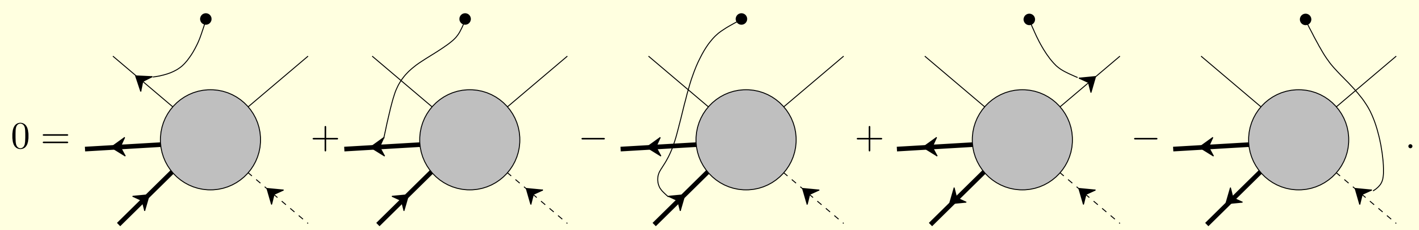
The tensorial index notation is cumbersome:

$$p'_a q'_b r'^c = G_a{}^f G_b{}^e G^c{}_d p_f q_e r^d$$

$$G_a{}^f G_b{}^e G^c{}_d = \delta_a^f \delta_b^e \delta_d^c + \epsilon_j ((T_j)_a^f \delta_b^e \delta_d^c + \delta_a^f (T_j)_b^e \delta_d^c - \delta_a^f \delta_b^e (T_j)_d^c) + O(\epsilon^2) \, ,$$

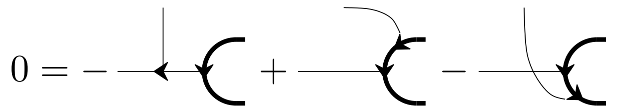
but diagramatically the group acts as a derivative (ingoing lines carry minus signs):

Invariance condition:

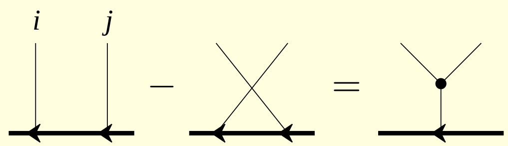


Lie algebra

As all other invariant tensors, the generators T_i must satisfy the invariance conditions:



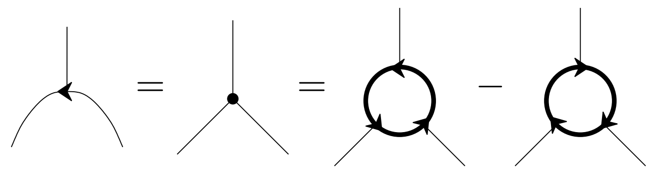
Redraw, replace the adjoint rep generators by the structure constants: we have derived the Lie algebra



$$T_i T_j \; - \; T_j T_i \; = \; iC_{ijk} T_k \, .$$

Structure constants

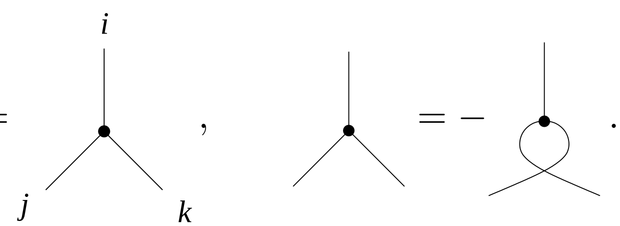
For a generator of an infinitesimal transformation acting on the adjoint rep, $A \to A$, it is convenient to replace the arrow by a full dot



$$(T_i)_{jk} \equiv -iC_{ijk} = -\mathrm{tr}\,[T_i, T_j]T_k\ ,$$

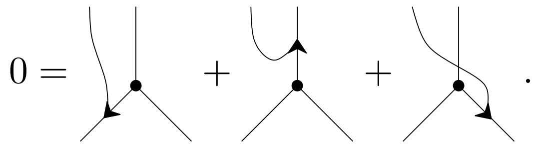
where dot stands for a fully antisymmetric structure constant iC_{ijk}. Keep track of the overall signs by always reading indices *counterclockwise* around a vertex

$-iC_{ijk} =$

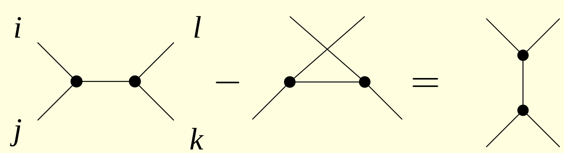


Jacobi relation

The invariance condition for structure constants C_{ijk} is likewise



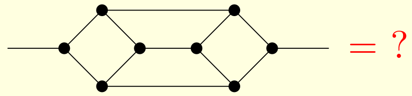
Rewdraw this with the dot-vertex to obtain the Jacobi relation



$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm} .$$

Birdtracks at work

Remember

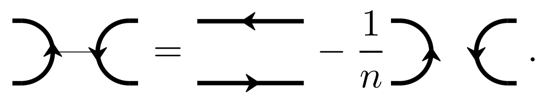


the one graph that launched this whole odyssey?

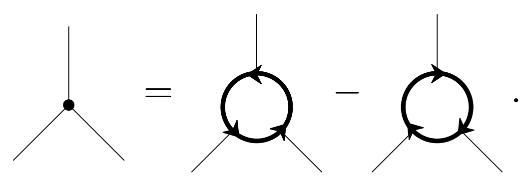
Example evaluation: $SU(n)$

We saw that the adjoint rep projection operators for the invariance group of the norm of a complex vector $|x|^2 = \delta_b^a x^b x_a$ is

$SU(n)$:



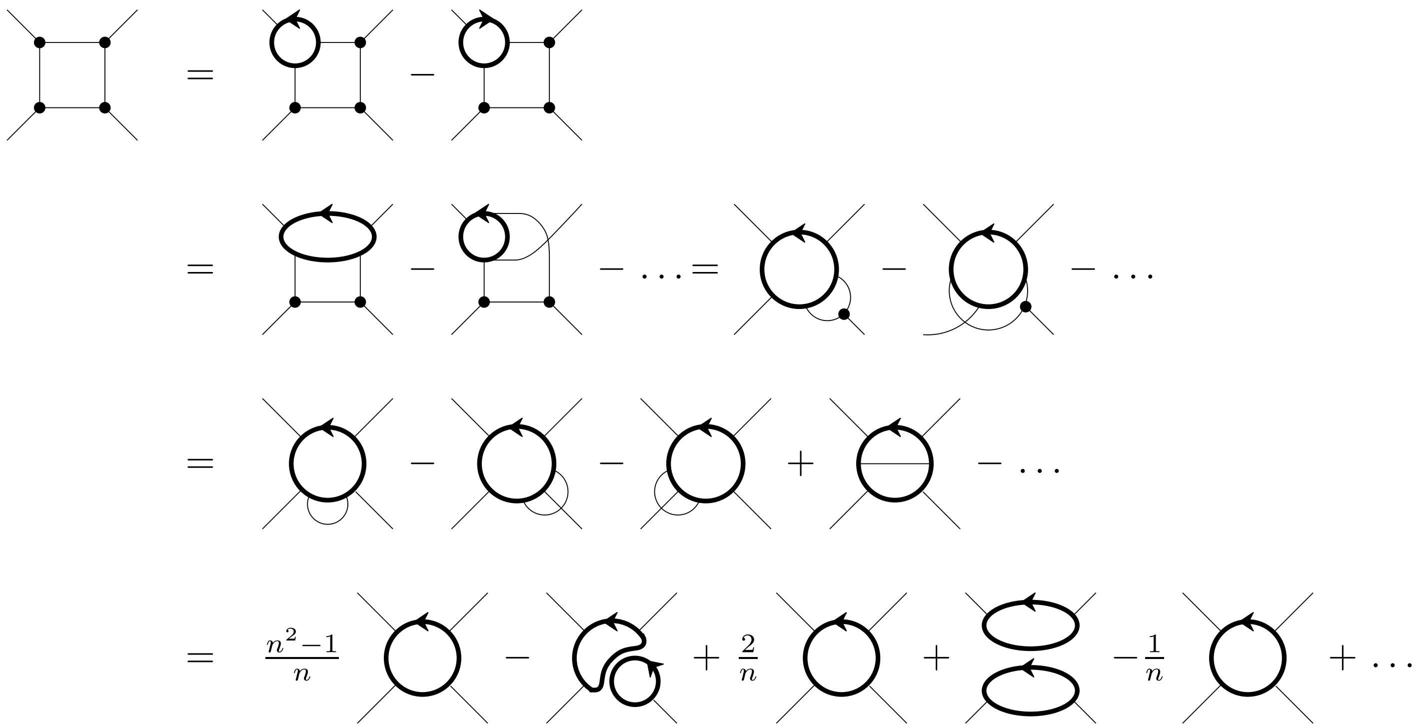
Eliminate C_{ijk} 3-vertices using



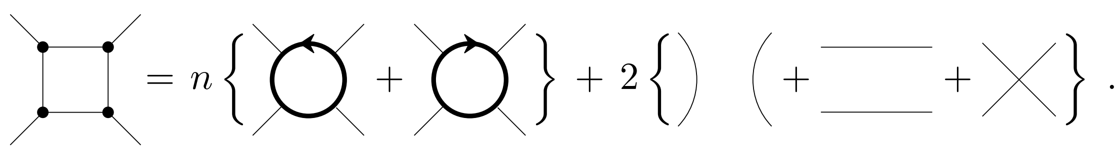
Heavy birdtracking, $SU(n)$

www.nbi.dk/GroupTheory

Evaluation is performed by a recursives substitution, the algorithm easily automated

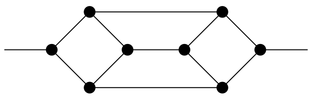


arriving at



$SU(n)$ 4-loop graph evaluated

Collecting everything together, we finally obtain

$SU(n)$:  $= 2n^2(n^2 + 12)$.

Any $SU(n)$ graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta_a^a = n$, the dimension of the defining rep.

A brief history of birdtracks

semi-simple Lie groups are here presented in an unconventional way, as “birdtracks”:

Wigner lineage:

1930: Wigner: all physics (atomic, nuclear, particle physics) = $3n$-j coefficients.

1956: I.B. Levinson: Wigner theory in graphical form (see A. P. Yutsis, I. Levinson and V. Vanagas, and G. E. Stedman).

Feynman lineage:

1949: R.P. Feynman: beautiful sketches of the very first “Feynman diagrams”

1971: R. Penrose’s drawings of symmetrizers and antisymmetrizers.

1974: G. ’t Hooft double-line notation for $U(n)$ gluons.

1976: P. Cvitanović[1,2] birdtracks for $SU(n)$, $SO(n)$ and $Sp(n)$; the exceptional Lie groups other than E_8.

[1] P. Cvitanović, *Phys. Rev.* **D14**, 1536 (1976)

[2] P. Cvitanović, Oxford preprint 40/77 (June 1977); `www.nbi.dk/ChaosBook`

Feynman diagrams? Why birdtracks?

Feynman diagrams are a memonic device, an aid in writing down an integral.

"Birdtracks" are a calculational method: here all calculations are carried out in terms of birtracks, from start to finish.

Part II: Exceptional magic

1. Lie groups as invariance groups
2. primitive invariants classification
3. $SU(n)$ as invariance group
4. E_6 family
5. G_2 family
6. E_8 family
7. Exceptional magic
8. Why did you do this?

Lie groups as invariance groups

i) define an invariance group by specifying a list of primitive invariants

ii) primitiveness and invariance conditions → algebraic relations between primitive invariants

iii) construct invariant matrices acting on tensor product spaces,

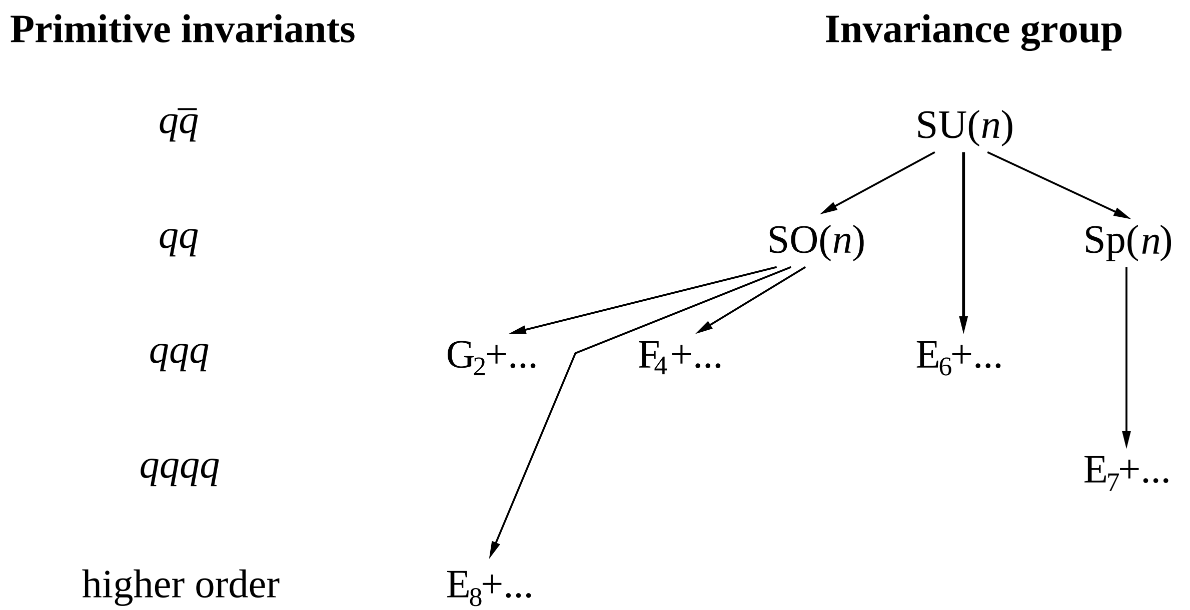
iv) construct projection operators for reduced rep from characteristic equations for invariant matrices.

When the next invariant is added, the group of invariance transformations of the previous invariants splits into two subsets; those transformations which preserve the new invariant, and those which do not.

Such decompositions yield Diophantine conditions on rep dimensions, so constraining that they limit the possibilities to a few which can be easily identified.

Classification by primitive invariants

The logic of the construction schematically indicated by the chains of subgroups



Example: E_7 primitives are:

a sesquilinear invariant $q\bar{q}$,
a skew symmetric qp invariant, and
a symmetric $qqqq$.

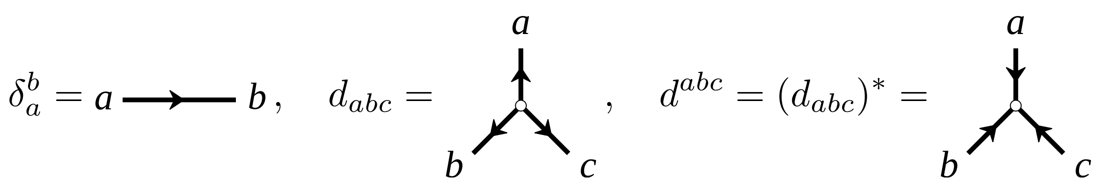
E_6 family of invariance groups

Example

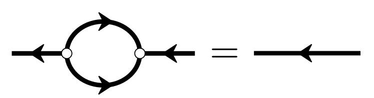
What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant

$$D(p,q,r) = D(q,p,r) = D(p,r,q) = d^{abc} p_a q_b r_c \;?$$

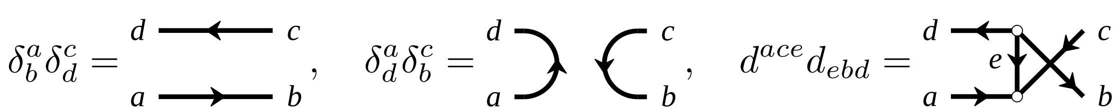
i) primitive invariant tensors:



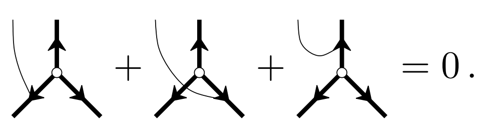
ii) primitiveness: $d_{aef} d^{efb}$ proportional to δ^a_b, the only primitive 2-index tensor. Fix the d_{abc}'s normalization:



iii) all invariant hermitian matrices in $V \otimes \bar{V} \to V \otimes \bar{V}$

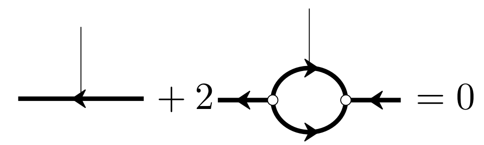


iv) invariance condition:

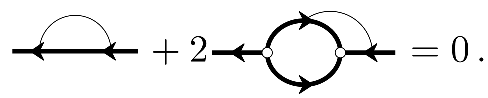


E_6 family: invariance condition

Contract the invariance condition with d^{abc} :

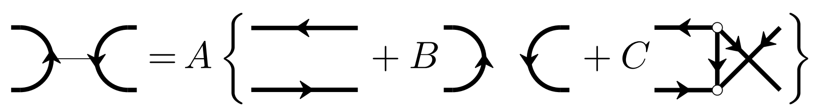


Contract with $(T_i)_a^b$ to get an invariance condition on the adjoint projection operator P_A:

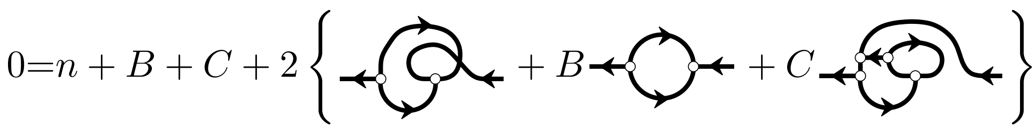


Adjoint projection operator in the invariant tensor basis (A, B, C to be fixed):

$$(T_i)_b^a (T_i)_c^d = A(\delta_c^a \delta_b^d + B\delta_b^a \delta_c^d + C d^{ade} d_{bce})$$



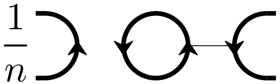
Substituting P_A



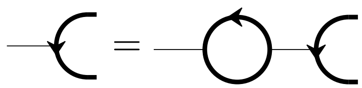
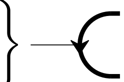
$$0 = B + C + \frac{n+2}{3} \,.$$

E_6 family: adjoint rep

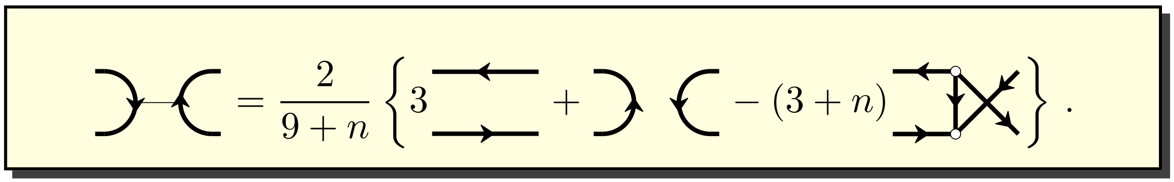
v) projection operators are orthonormal: P_A is orthogonal to the singlet projection operator P_1, $0 = P_A P_1$. This yields the second relation on the coefficients:

$0 = \frac{1}{n}$  $= 1 + nB + C$.

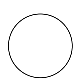
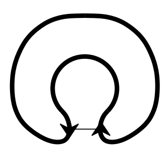
Normalization fixed by $P_A P_A = P_A$:

 $= A\left\{1 + 0 - \frac{C}{2}\right\}$  .

The 3 relations yield the adjoint projection operator



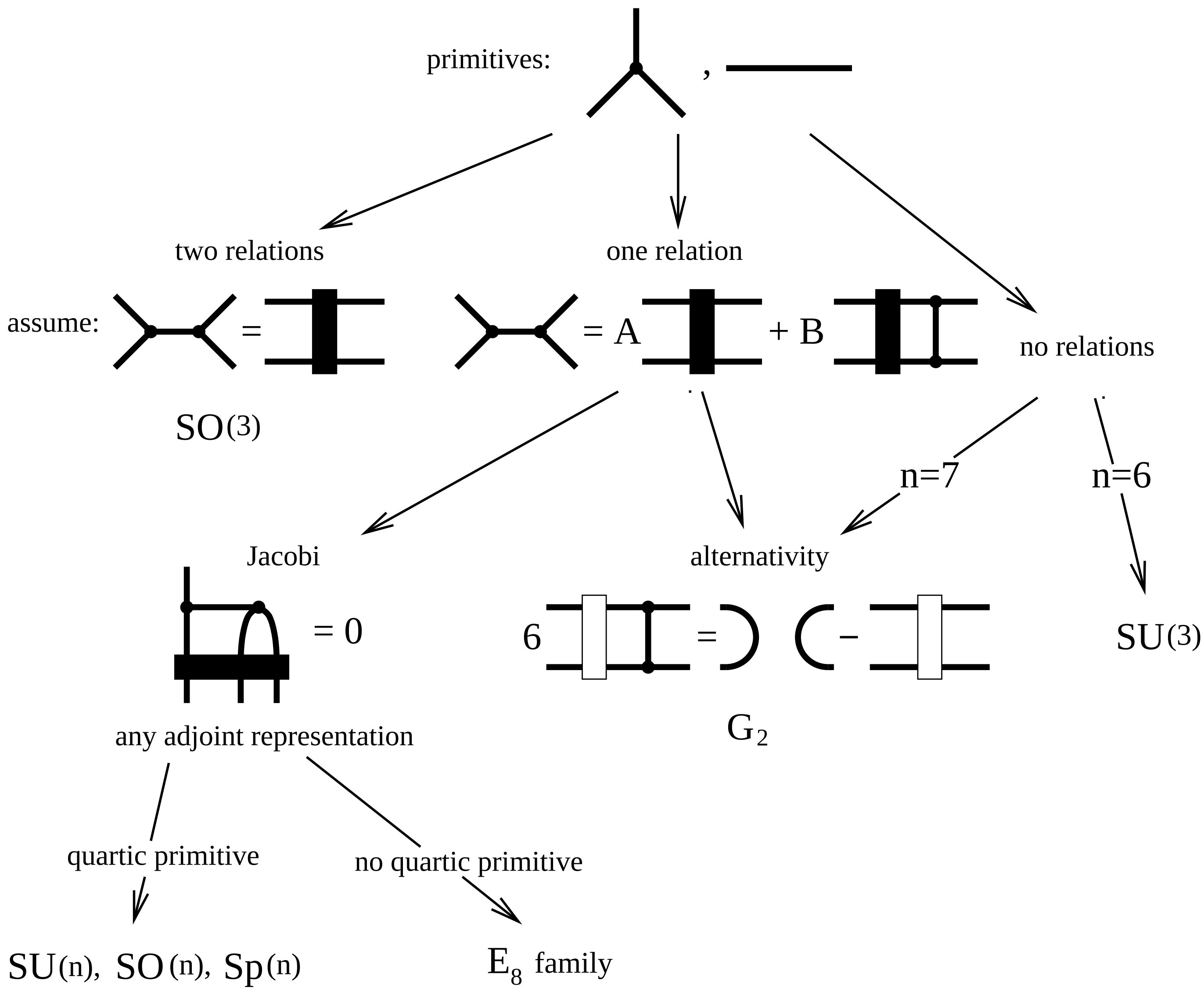
The dimension of the adjoint rep:

$N = \delta_{ii} =$  $=$  $= nA(n + B + C) = \frac{4n(n-1)}{n+9}$.

This Diophantine condition is satisfied by a small family of invariance groups, the E_6 row in the Magic Triangle, with E_6 corresponding to $n = 27$ and $N = 78$.

G_2 and E_8 families of invariance groups

www.nbi.dk/GroupTheory



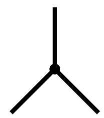
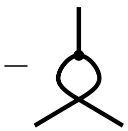
G_2 family of invariance groups

Primitive invariants:

(i) $\delta_b^a \rightarrow$ invariance group is a subgroup of $SU(n)$.

(ii) $\delta_{ab} \rightarrow$ invariance group is a subgroup of $SO(n)$.

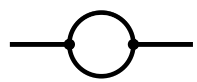
(iii) a cubic antisymmetric invariant

$f_{abc} =$  $= -$  $= -f_{acb}\,.$

Primitiveness assumption: all invariants are tree contractions of δ_{ab} , f_{abc}.

Example: the primitiveness assumption implies that

$$f_{abc}f_{cbd} = \alpha\, \delta_{ad}$$

 $= \alpha$  .

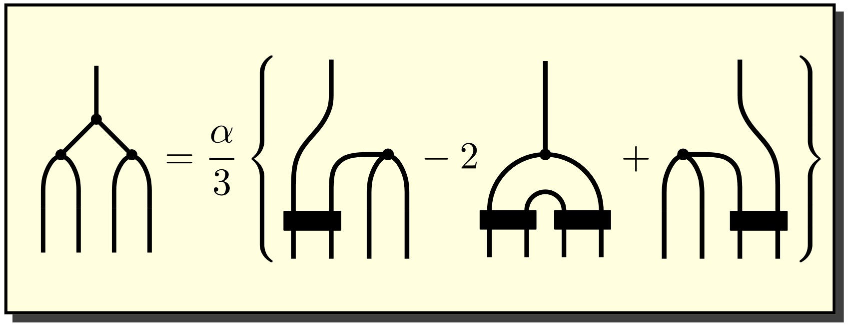
$\alpha = 1$ (normalization of f's) in what follows.

G_2 alternativity relation

Result: Invariance condition is nontrivially satisfied only in 3 and 7 dim - a proof of

Hurwitz's theorem: $n+1$ *dimensional normed algebras over reals exist only for* $n = 0, 1, 3, 7$ *(real, complex, quaternion, octonion).*

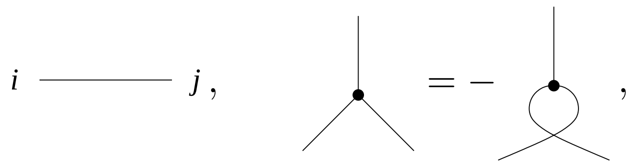
The full solution for G_2 is given by the reduction identity:



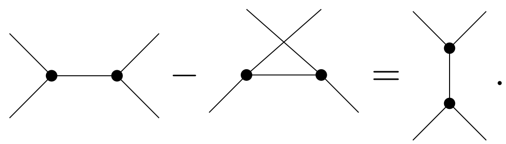
which recursively reduces all contractions of products of δ-functions and pairwise contractions $f_{abc}f_{cde}$, and thus completely solves the problem of evaluating any diagram of G_2.

E_8 family of invariance groups

primitives: symmetric quadratic, antisymmetric cubic primitive invariants:



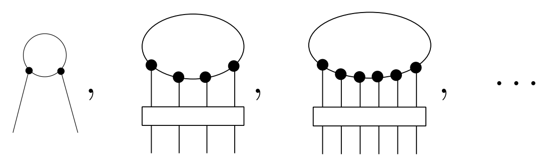
satisfying the **Jacobi relation:**



The task:

(i) enumerate all Lie groups that leave the primitives invariant.
The key idea here is the primitiveness assumption: any invariant tensor a linear sum over the tree invariants constructed from the quadratic and the cubic invariants, i.e. no quartic primitive invariant exists in the adjoint rep

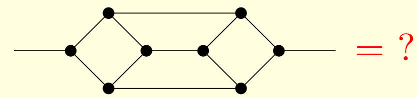
(ii) demonstrate that we can reduce all loops



(0.1)

E_8 family: Two-index tensors

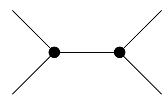
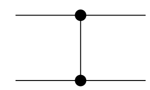
Remember

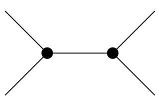


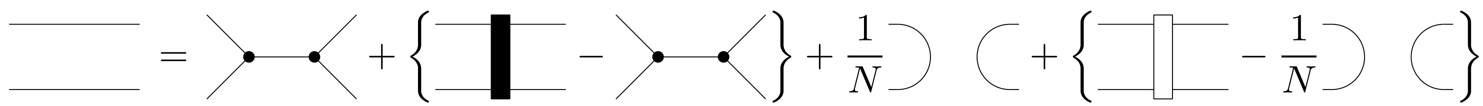
,

the one graph that launched this whole odyssey?

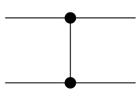
A loop with four structure constants is reduced by reducing the $A \otimes A \to A \otimes A$ space. By Jacobi relation there are only two linearly independent tree invariants in A^4 constructed from the cubic invariant:

 and 

 induces a decomposition of $\wedge^2 A$ antisymmetric tensors:



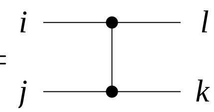
$$\mathbf{1} = \mathbf{P}_{\square} + \mathbf{P}_{\boxminus} + \mathbf{P}_{\bullet} + \mathbf{P}_{s} \,.$$

 matrix in $A \otimes A \to A \otimes A$ can decompose only the symmetric subspace $\mathrm{Sym}^2 A$.

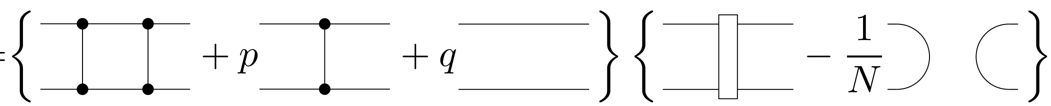
E_8 family: primitivness assumption

> The assumption that there exists no primitive quartic invariant is the defining relation for the E_8 family.

Let

$\mathbf{Q}_{ij,kl} =$  .

By the primitiveness assumption, the 4-index loop invariant $\mathbf{Q}^2$ is expressible in terms of $\mathbf{Q}_{ij,k\ell}$, $C_{ijm}C_{mk\ell}$ and δ_{ij}, hence on the traceless symmetric subspace

0= 

$$0=(\mathbf{Q}^2 + p\mathbf{Q} + q\mathbf{1})\mathbf{P}_s \,.$$

Coefficients p, q follow from symmetry and the Jacobi relation, yielding the characteristic equation for $\mathbf{Q}$

$$\left(\mathbf{Q}^2 - \frac{1}{6}\mathbf{Q} - \frac{5}{3(N+2)}\mathbf{1}\right)\mathbf{P}_s = (\mathbf{Q} - \lambda\mathbf{1})(\mathbf{Q} - \lambda^*\mathbf{1})\mathbf{P}_s = 0 \,.$$

Rewrite the condition on an eigenvalue of $\mathbf{Q}$

$$\lambda^2 - \frac{1}{6}\lambda - \frac{5}{3(N+2)} = 0 \,,$$

as fromula for N

$$N + 2 = \frac{5}{3\lambda(\lambda - 1/6)} = 60\left(\frac{6-\lambda^{-1}}{6} - 2 + \frac{6}{6-\lambda^{-1}}\right) \,.$$

As we shall seek for values of λ such that the adjoint rep dimension N is an integer, it is natural to reparametrize the two eigenvalues as

$$\lambda = \frac{1}{6}\frac{1}{1-m/6} = -\frac{1}{m-6}\,, \qquad \lambda^* = \frac{1}{6}\frac{1}{1-6/m} = \frac{1}{6}\frac{m}{m-6}\,.$$

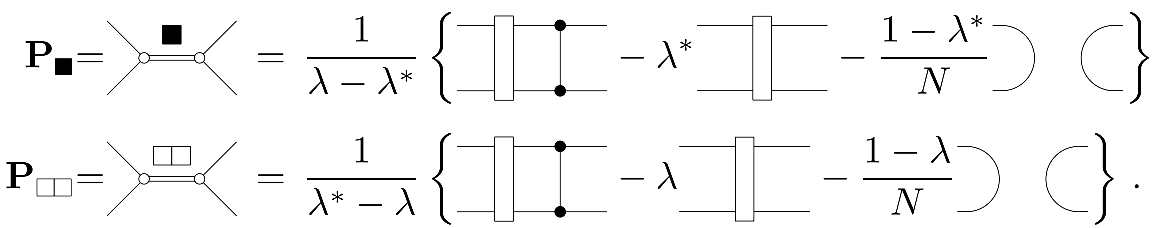
In terms of the parameter m, the dimension of the adjoint representation is given by

$$N = -122 + 10m + 360/m\,.$$

As N is an integer, allowed m are rationals $m = P/Q$, P and Q relative primes. Need to check only the 27 rationals $m > 6$.

E_8 family: further Diphantine conditions

The associated projection operators:



Typical dimensions:

$$d_{\square\square} = \mathrm{tr}\,\mathbf{P}_{\square\square} = \frac{(N+2)(1/\lambda+N-1)}{2(1-\lambda^*/\lambda)} = \frac{5(m-6)^2(5m-36)(2m-9)}{m(m+6)},$$

$$d_{\blacksquare} = \frac{270(m-6)^2(m-5)(m-8)}{m^2(m+6)}.$$

From the decomposition of the $\mathrm{Sym}^3 A$:

$$d_{\square} = \frac{5(m-5)(m-8)(m-6)^2(2\,m-15)(5\,m-36)}{m^3(3+m)(6+m)}(36-m)$$

To summarize: $A \otimes A$ decomposes into 5 irreducible reps

$$\mathbf{1} = \mathbf{P}_{\square} + \mathbf{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \mathbf{P}_{\bullet} + \mathbf{P}_{\square\square} + \mathbf{P}_{\blacksquare}.$$

The decomposition is parametrized by a rational m and is possible only if dimensions N and $d_{\square\square}$ are integers. our homework problem is done: a reduction of the adjoint rep 4-vertex box for all exceptional Lie groups. The main result of all this heavy birdtracking: $N > 248$ is excluded by the positivity of $d_{\blacksquare}$, $N = 248$ is special, as $\mathbf{P}_{\blacksquare} = 0$ implies existence of a tensorial identity on the $\mathrm{Sym}^3 A$ subspace.

E_8 family: Diophantine conditions

The $A \otimes A \to A \otimes A$ Diophantine conditions are satisfied only for

m	5	8	9	10	12	15	18	24	30	36
N	0	3	8	14	28	52	78	133	190	248
d_5	0	0	1	7	56	273	650	1,463	1,520	0
$d_{\square}$	0	-3	0	64	700	4,096	11,648	40,755	87,040	147,250
$d_{\square\blacksquare}$	0	0	27	189	1,701	10,829	34,749	152,152	392,445	779,247

I eliminate (indirectly) $m = 30$ by the semi-simplicity condition. J. M. Landsberg and L. Manivel[1] identify the $m = 30$ solution as a non-reductive Lie algebra.

[1] J. M. Landsberg and L. Manivel, *Advances in Mathematics* **171**, 59-85 (2002); `arXiv:math.AG/0107032, 2001`

Exceptional magic

A closer scrutiny of the solutions (column,row) = $(m, l) \in \{8, 9, 10, 12, 15, 18, 24, 30, 36\}$ to all $V \otimes \bar{V} \to V \otimes \bar{V}$ Diophantine conditions

m	8	9	10	12	15	18	20	24	30	36	40	$\cdots$	360
F_4			0	0	3	8	.	21	.	52	.	$\cdots$	.
E_6		0	0	2	8	16	.	35	36	78	.	$\cdots$	.
E_7	0	1	3	9	21	35	.	66	99	133	.	$\cdots$	.
E_8	3	8	14	28	52	78	.	133	190	248	.	$\cdots$	.

leads to a surprise: all of them are the one and the same condition

$$N = \frac{(\ell - 6)(m - 6)}{3} - 72 + \frac{360}{\ell} + \frac{360}{m}$$

magically arrange all exceptional families into a Magic Triangle.
All $A \otimes V$ Kronecker product characteristic equations are also of the same form

$$(\mathbf{Q} - 1)\,(\mathbf{Q} + 6/m)\,P_r = 0\,.$$

J. M. Landsberg and L. Manivel[1] identify the $m = 30$ column as a non-reductive Lie algebra.

[1] J. M. Landsberg and L. Manivel, *Advances in Mathematics* **171**, 59-85 (2002); `arXiv:math.AG/0107032, 2001`

Magic Triangle

www.nbi.dk/GroupTheory

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $3\mathbf{A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $2\mathbf{A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $3\mathbf{A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Magic triangle: All solutions of the Diophantine conditions place the defining and adjoint reps exceptional Lie groups into a triangular array. Within each entry: the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining rep.

The expressions for n for the top four rows are guesses. The triangle is called “magic”, because it contains the Freudenthal’s Magic Square.

A brief history of exceptional magic

1975-77: Primitive invariants construction of all semi-simple Lie algebras[1,2], except for the E_8 family.

1979: E_8 family primitivness assumption (no quartic primitive invariant), inspired by Okubo's observation[3] that the quartic Dynkin index vanishes for the exceptional Lie algebras.

1981: Magic Triangle, the E_7 family and its $SO(4)$-family of "negative dimensional" relatives derived and discussed in detail[4]. The total number of citations in the next 22 years: 2 (two).

1987(?)-2001: Angelopoulos[5] classifies Lie algebras by the spectrum of the Casimir operator acting on $A \otimes A$, and, *inter alia*, obtains the same E_8 family.

1995 : Vogel[6] notes that for the exceptional groups the dimensions and casimirs of the $A \otimes A$ adjoint rep tensor product decomposition $\mathbf{P}_{\square} + \mathbf{P}_{\substack{\square\\\square}} + \mathbf{P}_{\bullet} + \mathbf{P}_{\square\square} + \mathbf{P}_{\blacksquare}$ are rational functions of parameter a (related to my parameter m by $a = 1/m - 6$.)

1996: Deligne[7] conjectures that for A_1, A_2, G_2, F_4, E_6, E_7 and E_8 the dimensions of higher tensor reps $\otimes A^k$ could likewise be expressed as rational functions of parameter a.

1996: Cohen and de Man[8] computer verifications of the Deligne conjecture for all reps up to $\otimes A^4$. They

[1] P. Cvitanović, *Phys. Rev.* **D14**, 1536 (1976)
[2] P. Cvitanović, Oxford preprint 40/77 (June 1977); `www.nbi.dk/ChaosBook`
[3] S. Okubo, *J. Math. Phys.* **20**, 586 (1979)
[4] P. Cvitanović, *Nucl. Phys.* **B188**, 373 (1981)
[5] E. Angelopoulos, *Panamerican Math. Jour.* **2**, 65-79 (2001)
[6] P. Vogel, "Algebraic structures on modules of diagrams," preprint (1995)
[7] P. Deligne, *C.R. Acad. Sci. Paris, Sér. I,* **322**, 321 (1996)
[8] A. M. Cohen and R. de Man, *C.R. Acad. Sci. Paris, Ser. I,* **322**, 427 (1996)

note that "miraculously for all these rational functions both numerator and denominator factor in $Q[a]$ as a product of linear factors". (This is immediate in my derivation)

1999: Cohen and de Man[9] derive the same projection operators and dimension formulas by the same birdtrack computations for the E_8 family (do refer to my webbook, not noticing that the calculation is already there).

2001-2003: J. M. Landsberg and L. Manivel[10] utilise projective geometry and triality to interpret the Magic Triangle, recover the known dimension and decomposition formulas, and derive an infinity of higher-dimensional rep formulas.

2002: Deligne and Gross[11] (re)discover the Magic Triangle.

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[9] A. M. Cohen and R. de Man, in P. Drexler *et al.*, *Progress in Math.* **173, *Euroconf. Proceedings*** (Birkhäuser, Basel, 1999)
[10] J. M. Landsberg and L. Manivel, *Advances in Mathematics* **171**, 59-85 (2002); `arXiv:math.AG/0107032, 2001`
[11] P. Deligne and B. H. Gross, *C.R. Acad. Sci. Paris, Sér. I,* **335**, 2002 (2002)

Epilogue

"Why did you do this?" you might well ask.
OK, here is an answer.

It has to do with a conjecture of finiteness of gauge theories, which, by its own twisted logic, led to this sidetrack, birdtracks and exceptional Lie algebras:

If gauge invariance of QED guarantees that all UV and IR divergences cancel, why not also the finite parts?

And indeed; when electron magnetic moment diagrams are grouped into gauge invariant subsets, a rather surprising thing happens[1]; while the finite part of each Feynman diagram is of order of 10 to 100, every subset computed so far adds up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n .$$

If you take this numerical observation seriously, the "zeroth" order approximation to the electron magnetic moment is given by

$$\frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{"corrections"}.$$

Now, this is a great heresy - my colleagues will tell you that Dyson has shown that the perturbation expansion is an asymptotic series, in the sense that the nth order contribution should be exploding combinatorially

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

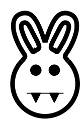
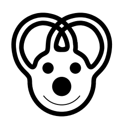
[1] P. Cvitanović, "Asymptotic estimates and gauge invariance," *Nucl. Phys.* **B127**, 176 (1977)

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + n\left(\frac{\alpha}{\pi}\right)^n + \cdots .$$

I kept looking for a simpler gauge theory in which I could compute many orders in perturbation theory and check the conjecture. We learned how to count Feynman diagrams. I formulated a planar field theory whose perturbation expansion is convergent. I learned how to compute the group weights of Feynman diagrams in non-Abelian gauge theories. By marrying Poincaré to Feynman we found a new perturbative expansion more compact than the standard Feynman diagram expansions. No dice. To this day I still do not know how to prove or disprove the conjecture.

QCD quarks are supposed to come in three colors. This requires evaluation of SU(3) group theoretic factors, something anyone can do. In the spirit of Teutonic completeness, I wanted to check all possible cases; what would happen if the nucleon consisted of 4 quarks, doodling

 $-$  $= n(n^2-1)$,

and so on, and so forth. In no time, and totally unexpectedly, all exceptional Lie groups arose, not from conditions on Cartan lattices, but on the same geometrical footing as the classical invariance groups of quadratic norms, $SO(n)$, $SU(n)$ and $Sp(n)$.

Magic ahead

Nobody, but truly nobody in those days showed a glimmer of interest in the exceptional Lie algebra parts of this work, so there was no pressure to publish it before completing it:

by completing it I mean finding the algorithms that would reduce any bubble diagram to a number for any semi-simple Lie algebra. The task is accomplished for G_2, but for F_4, E_6, E_7 and E_8 this is still an open problem. This, perhaps, is only matter of algebra (all of my computations were done by hand, mostly on trains and in airports), but the truly frustrating unanswered question is:

Where does the Magic Triangle come from? Why is it symmetric across the diagonal? Something is happening here, but my derivation misses it. Most likely the starting idea - to classify all simple Lie groups from the primitivness assumption - is flawed. Is there a mother of all Lie algebras, some complex function which yields the Magic Triangle for a set of integer values?

And then there is a practical issue of unorthodox notation: transferring birdtracks from hand drawings to LaTeX took another 21 years. In this I was rescued by Anders Johansen who undertook drawing some 4,000 birdtracks needed to complete this manuscript, of elegance far outstriping that of the old masters.