In honor of

Freeman Dyson's

80th birthday

www.nbi.dk/GroupTheory

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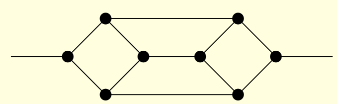
Q : What is the group theoretic weight for QCD diagram (asymptotic freedom?)



$= ?$

A :

1. new notation: invariant tensors $\leftrightarrow$ "Feynman" diagrams
2. new computational method: diagrammatic, start $\longrightarrow$ finish
3. new relations: "negative dimensions" $SO(n) \leftrightarrow Sp(-n)$, $E_7 \leftrightarrow SO(4)$, etc.
4. new classification: primitive invariants $\rightarrow$ all semi-simple Lie algebras

Magic Triangle

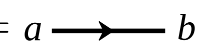
						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Part I: $SU(n)$, $SO(n)$, $Sp(n)$: a review

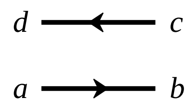
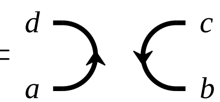
1. invariance groups of quadratic norms
2. birdtrack notation
3. reduction of multi-particle states

$U(n)$ invariant matrices

$U(n)$: invariance group of the norm of a complex vector $|x|^2 = \delta_b^a x^b x_a$.

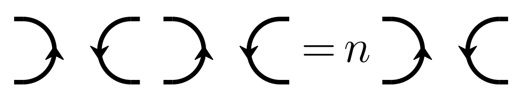
only primitive invariant tensor: $\delta_b^a =$ 

2 invariant tensors $M \in V^2 \otimes \bar{V}^2$:

identity : $\mathbf{1}_{d,b}^{a\,c} = \delta_b^a \delta_d^c =$  , trace : $T_{d,b}^{a\,c} = \delta_d^a \delta_b^c =$  .

Evaluation of T^2 in tensor, birdtrack, matrix notation:

$$T_{d,e}^{a\,f} T_{f,b}^{e\,c} = \delta_d^a \delta_e^f \delta_f^e \delta_b^c = n\, T_{d,b}^{a\,c}\,,$$



$$T^2 = nT\,.$$

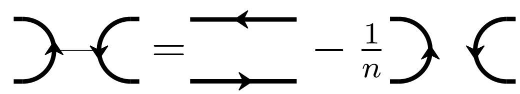
where

$$\delta_e^e = n = \text{the dimension of the defining vector space } V$$

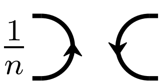
$U(n)$ reduction

Trace + traceless projection operators decompose $U(n) \to SU(n) \oplus U(1)$:

$SU(n)$ adjoint rep: $P_1 = \mathbf{1} - \frac{1}{n}T$



$U(n)$ singlet: $P_2 = \frac{1}{n}T =$ $\frac{1}{n}$



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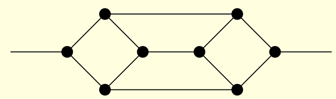
Birdtracks at work

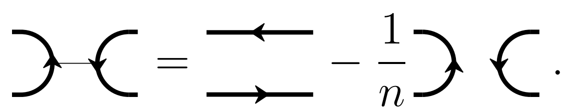
Example: $SU(n)$ evaluation of



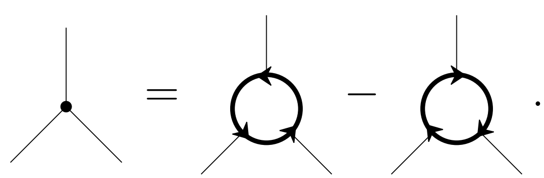
$= ?$,

The adjoint rep (all traceless matrices) projection operator

$SU(n)$:

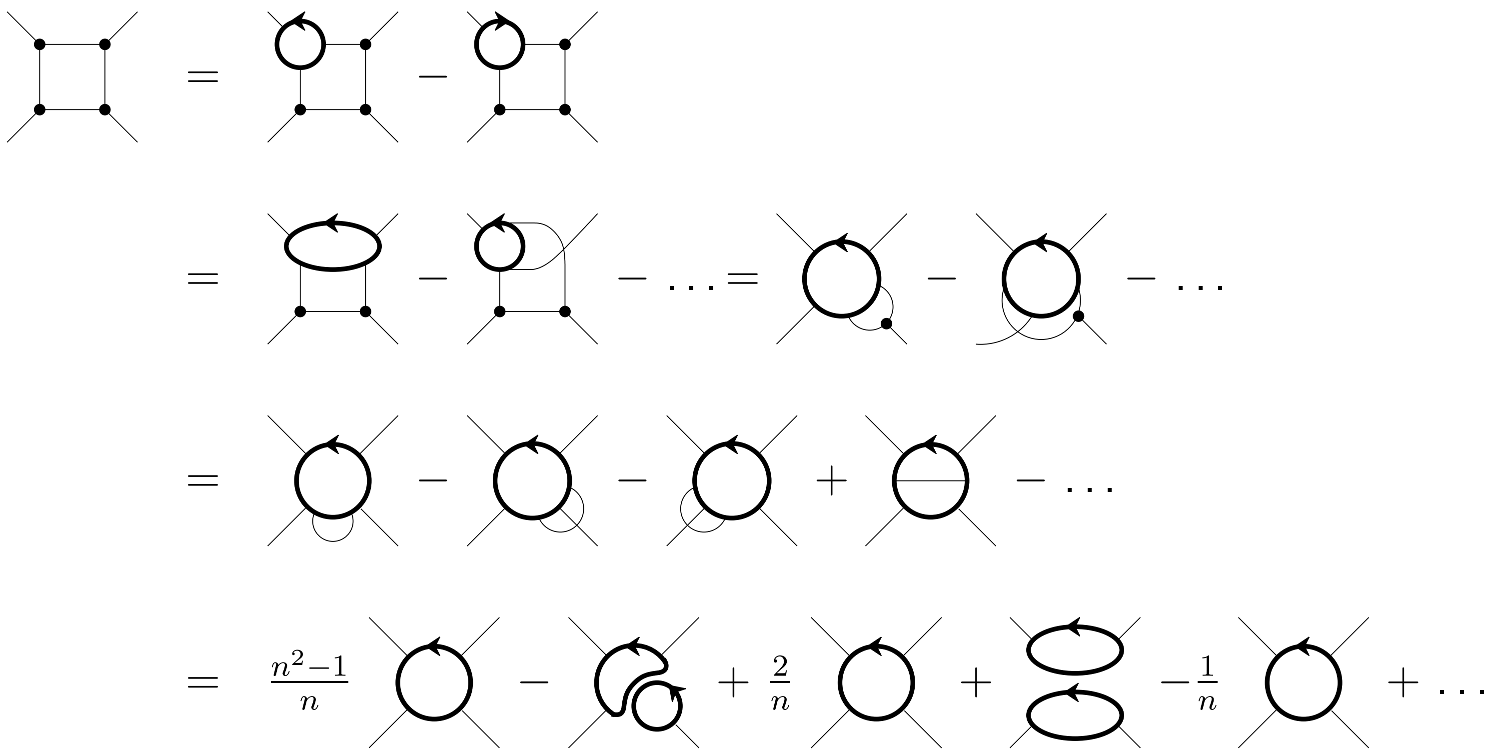


Eliminate structure constant C_{ijk} 3-vertices using

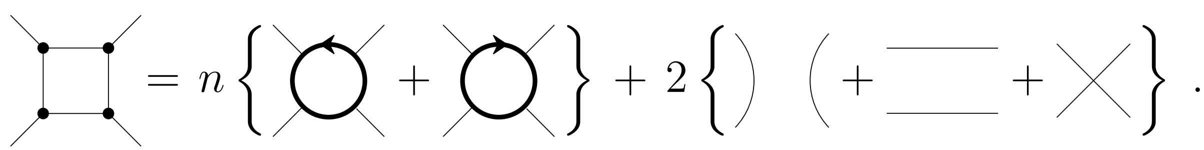


Heavy birdtracking, $SU(n)$

Evaluation is performed by a recursive substitution, the algorithm easily automated

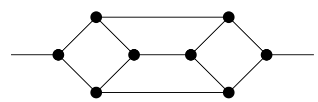


arriving at



$SU(n)$ 4-loop graph evaluated

Collecting everything together, we finally obtain

$SU(n)$:  $= 2n^2(n^2 + 12)$ ——— .

Any $SU(n)$ graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta_a^a = n$, the dimension of the defining rep.

A brief history of birdtracks

Wigner lineage:

1930: Wigner: all physics (atomic, nuclear, particle physics) $= 3n\text{-}j$ coefficients.

1956: I.B. Levinson: Wigner theory in graphical form (see A. P. Yutsis, I. Levinson and V. Vanagas, and G. E. Stedman).

Feynman lineage:

1949: R.P. Feynman: beautiful sketches of the very first "Feynman diagrams"

1971: R. Penrose's drawings of symmetrizers and antisymmetrizers.

1974: G. 't Hooft double-line notation for $U(n)$ gluons.

1976: P. Cvitanović[1,2] birdtracks for $SU(n)$, $SO(n)$ and $Sp(n)$; the exceptional Lie groups other than E_8.

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[1] P. Cvitanović, *Phys. Rev.* **D14**, 1536 (1976)

[2] P. Cvitanović, Oxford preprint 40/77 (June 1977); www.nbi.dk/ChaosBook

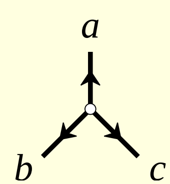
Cubic and higher invariants?

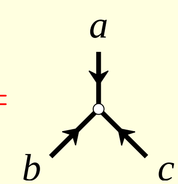
Suppose someone came into your office and asked

Q :

> "On planet **Z**, mesons consist of quarks and antiquarks, but baryons contain 3 quarks in a symmetric color combination. What is the color group?"

invariant tensors:

$\delta_a^b = a \longrightarrow b$, $d_{abc} =$ 

, $d^{abc} = (d_{abc})^* =$ 

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A :

> neither trivial, nor without beauty:
>
> On planet **Z** quarks can come in 27 colors, and the color group can be the exceptional E_6.

(No Killing-Cartan anywhere)

Part II: Invariance groups, a prelude

1. invariance groups
2. primitive invariants
3. reduction of multi-particle states

Invariants, invariance groups

Generalize length $q^2 = \delta_{ab} q_b q_a$ to cubic and higer invariants:

$$p(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c} \, q^a r^b \ldots s_c$$

is an invariant of the group $\mathcal{G}$ if for all $G \in \mathcal{G}$ and any set of vectors $q, r, s, \ldots$ it satisfies

invariance condition: $\quad p(\overline{Gq}, \overline{Gr}, \ldots Gs) = p(\overline{q}, \overline{r}, \ldots, s)\,.$

Definition. An invariance group $\mathcal{G}$ is the set of all linear transformations which leave invariant

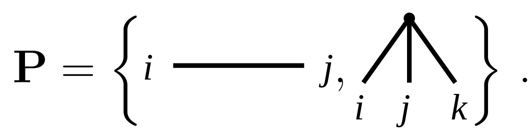
$$p_1(x, \bar{y}) = p_1(Gx, \bar{y}G^\dagger)\,, \qquad p_2(x, y, z, \ldots) = p_2(Gx, Gy, Gz \ldots)\,, \qquad \ldots\,.$$

a **finite** list of primitive invariants.

Primitive invariants

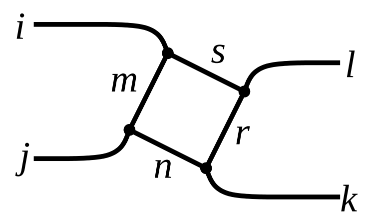
Definition. An invariant tensor is primitive if it cannot be expressed as a combination of tree invariants composed of other primitive invariant tensors.

Example:
Kronecker delta and Levi-Civita tensor are the primitive invariant tensors of our 3-dimensional space.

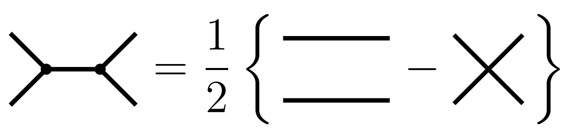


For $SO(3)$ the 4-vertex loop

$$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} =$$



with interal loop indices m, n, r, s summed over, is not a primitive, because the Levi-Civita relation



reduces it to a sum of tree invariant tensors.

Invariance, infinitesimally

Invariace of tensor h under infinitesimal $G : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$:

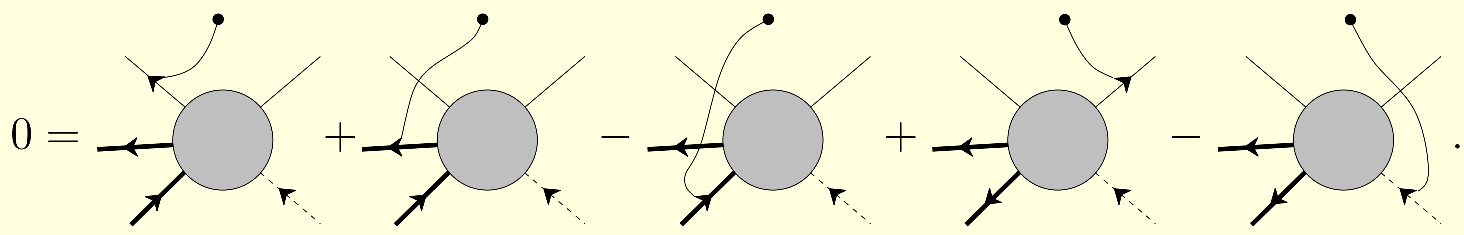
$$G_\alpha{}^\beta h_\beta = (\delta_\alpha{}^\beta + \epsilon_j (T_j)_\alpha{}^\beta) h_\beta + O(\epsilon^2) = h_\alpha \,.$$

Generators of infinitesimal transformations annihilate invariant tensors

$$T_i h = 0 \,.$$

Diagramatically, a derivative:

Invariance condition:



Lie algebra, Jacobi relation

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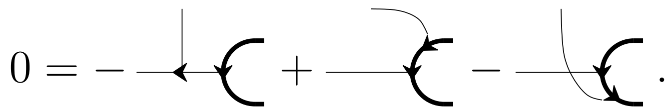
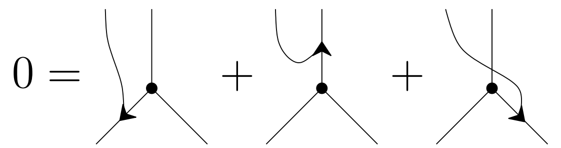
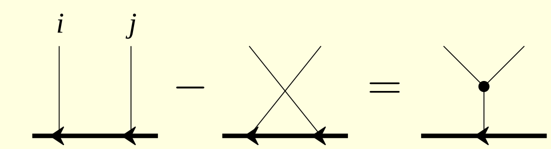
Example: The generators T_i and the structure constants C_{ijk} are invariant tensors:

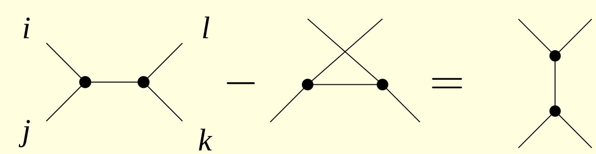




Rewdraw, obtain the Lie algebra and the Jacobi relation



$$T_iT_j - T_jT_i = iC_{ijk}T_k\,.$$



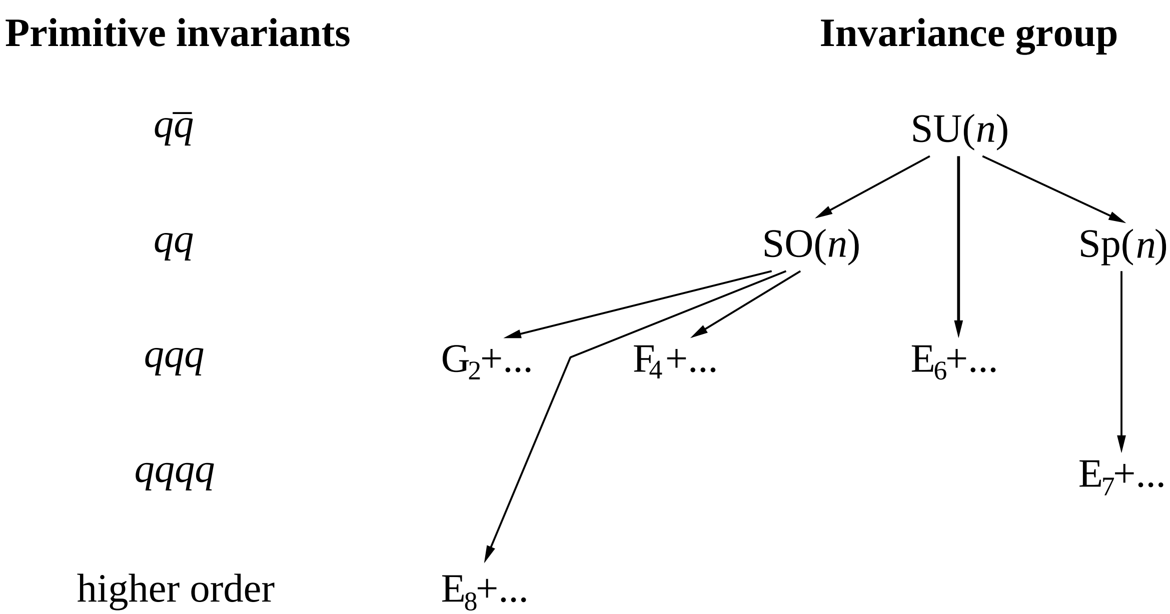
$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}\,.$$

Part III: Exceptional magic

1. primitive invariants classification
2. E_8 family
3. exceptional magic
4. why did you do this?

Invariance grups; classification

Strategy:



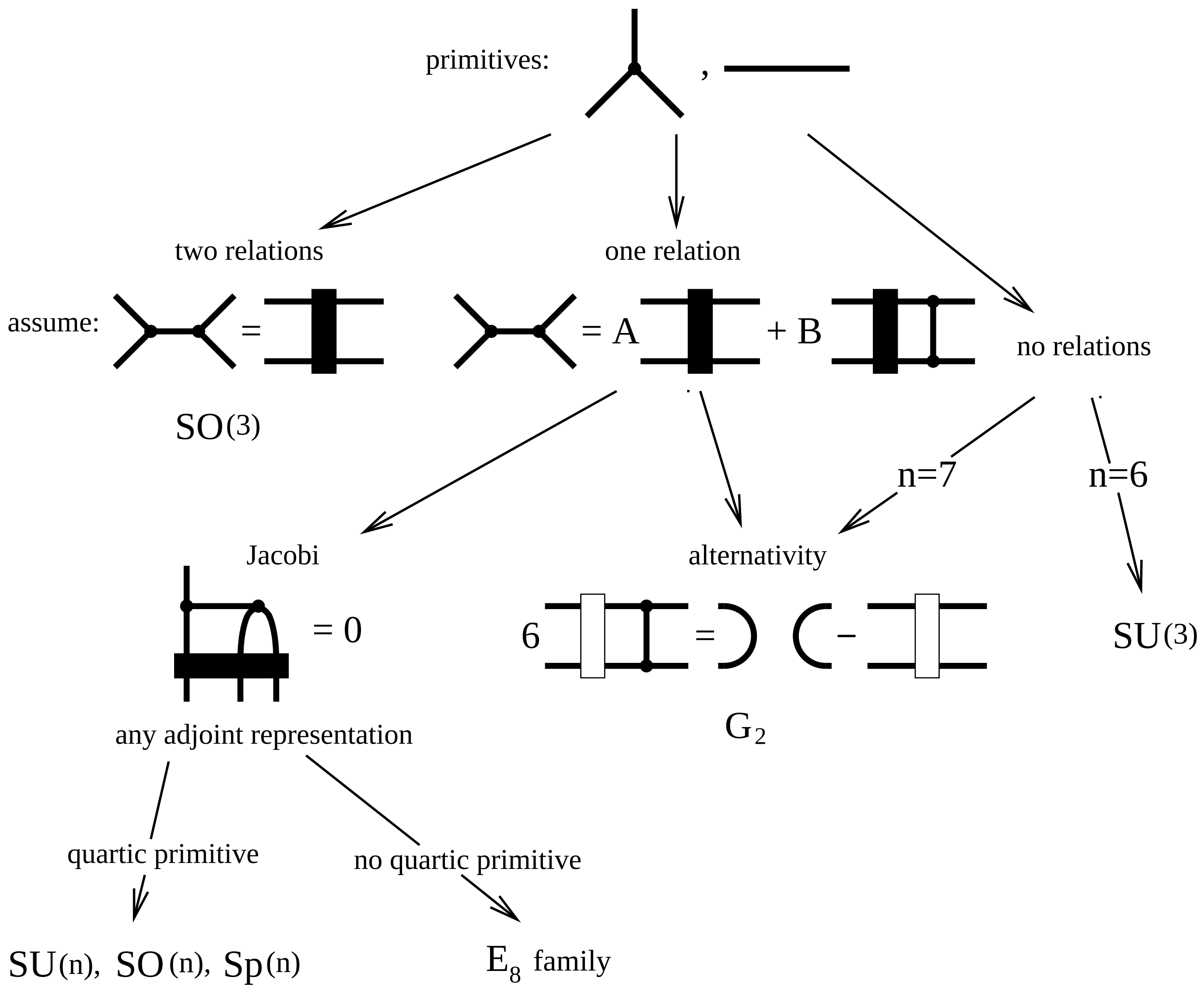
Example: E_7 primitives are:

a sesquilinear invariant $q\bar{q}$,

a skew symmetric qp invariant, and

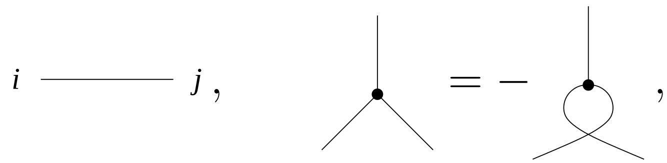
a symmetric $qqqq$.

G_2 and E_8 families of invariance groups

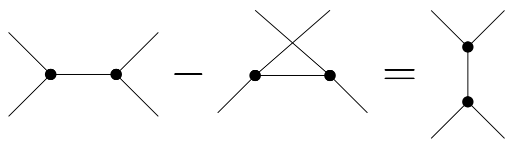


E_8 family of invariance groups

primitives: symmetric quadratic, antisymmetric cubic primitive invariants:

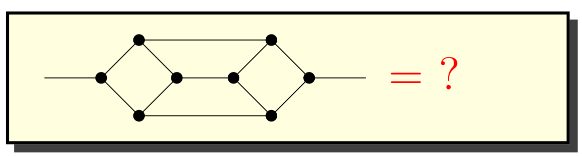


satisfying the **Jacobi relation:**



No quartic primitive invariant exists: Any invariant tensor a linear sum over the tree invariants constructed from the quadratic and the cubic invariants,

Remember

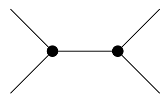
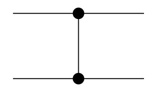


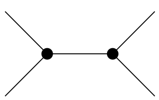
,

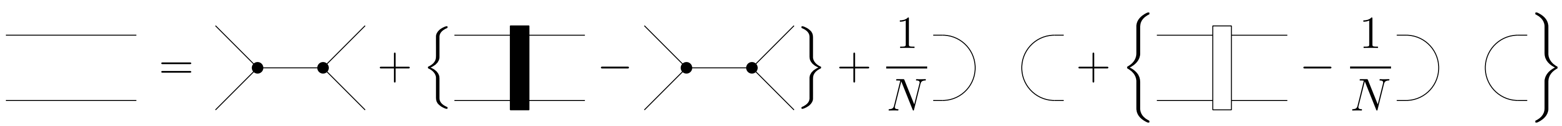
the one graph that launched this whole odyssey?

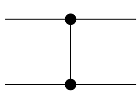
E_8 family: Two-index tensors

Jacobi relation: only two linearly independent tree invariants in $A \otimes A \rightarrow A \otimes A$ constructed from the cubic invariant:

 and 

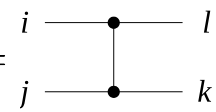
 induces a decomposition of $\wedge^2 A$ antisymmetric tensors:

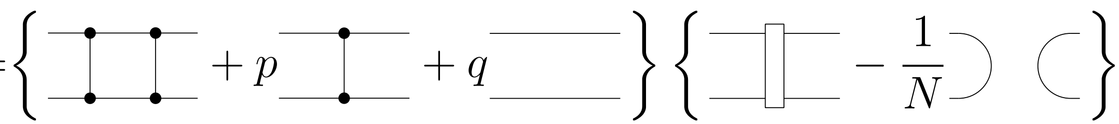


 matrix in $A \otimes A \rightarrow A \otimes A$ can decompose only the symmetric subspace $\mathrm{Sym}^2 A$.

E_8 family: primitiveness assumption

The assumption that there exists no primitive quartic invariant is the defining relation for the E_8 family.

4-index loop invariant $\mathbf{Q}^2$ is expressible in terms of $\mathbf{Q}_{ij,k\ell} =$ , $C_{ijm}C_{mk\ell}$ and δ_{ij}, splits the traceless symmetric subspace into 2 irreps

$0=$ 

$$0 = (\mathbf{Q}^2 + p\mathbf{Q} + q\mathbf{1})\mathbf{P}_s \,.$$

Symmetry, the Jacobi relation:

$$\left(\mathbf{Q}^2 - \frac{1}{6}\mathbf{Q} - \frac{5}{3(N+2)}\mathbf{1}\right)\mathbf{P}_s = (\mathbf{Q} - \lambda\mathbf{1})(\mathbf{Q} - \lambda^*\mathbf{1})\mathbf{P}_s = 0 \,.$$

so the two eigenvalues (of the quadratic Casimir operator) related to N by

$$\lambda^2 - \frac{1}{6}\lambda - \frac{5}{3(N+2)} = 0 \,.$$

E_8 family: quadratic Casimir

Eigenvalue (quadratic Casimir operator) satisfies

$$\lambda^2 - \frac{1}{6}\lambda - \frac{5}{3(N+2)} = 0\,,$$

Lie group dimension N is an integer: a convenient reparametrization

$$\lambda = -\frac{1}{m-6}$$

yields a Diphantine condition on the parameter m (*i.e.*, the quadratic Casimir):

$$N = -122 + 10m + 360/m\,.$$

keep computing ...

E_8 family: Diphantine conditions

$A \otimes A$ decomposes into 5 irreducible reps. $\mathrm{Sym}^3 A$ decomposed likewise, "after some algebra". The dimension formulas for irreps yield a bevy of Diphantine conditions:

$$N = -122 + 10m + 360/m\,.$$

$$d_{\square\square} = \frac{5(m-6)^2(5m-36)(2m-9)}{m(m+6)}\,,$$

$$d_{\blacksquare} = \frac{270(m-6)^2(m-5)(m-8)}{m^2(m+6)}\,.$$

$$d_{\square} = \frac{5(m-5)(m-8)(m-6)^2(2\,m-15)(5\,m-36)}{m^3(3+m)(6+m)}(36-m)$$

Our homework problem **done**: a reduction of the adjoint rep 4-vertex box for all exceptional Lie groups.

E_8 family: Diophantine conditions

All solutions of the (known) Diophantine conditions only 9 solutions (!):

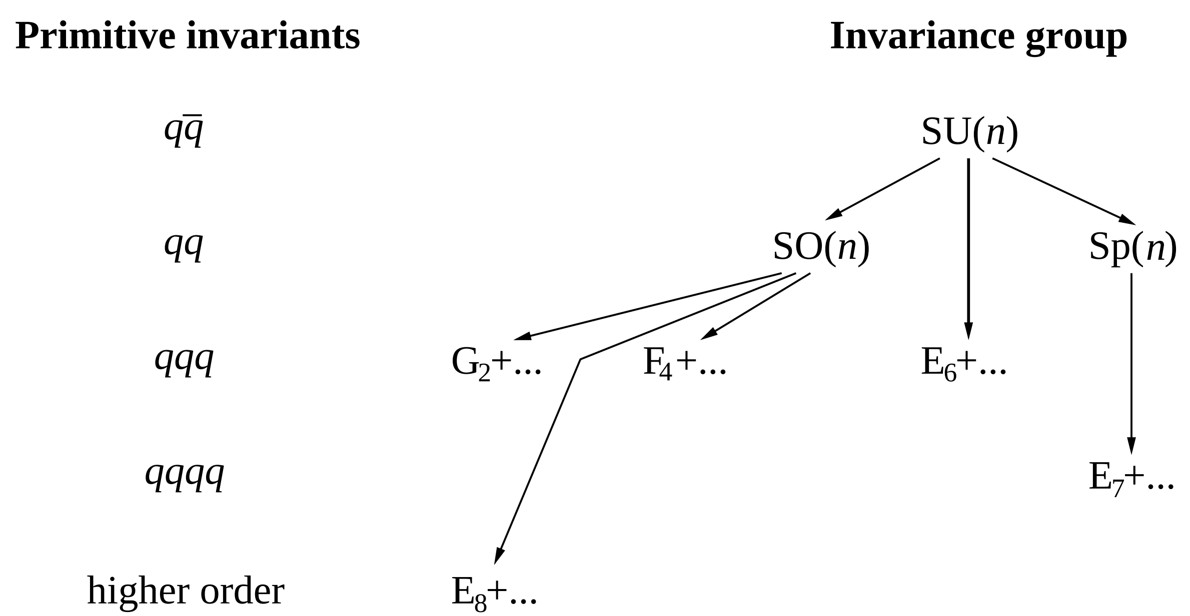
m	5	8	9	10	12	15	18	24	36
N	0	3	8	14	28	52	78	133	248
d_5	0	0	1	7	56	273	650	1,463	0
$d_{\square}$	0	-3	0	64	700	4,096	11,648	40,755	147,250
$d_{\square\blacksquare}$	0	0	27	189	1,701	10,829	34,749	152,152	779,247

E_8 248-dim representation, plus all exceptional Lie algebras, in one family!

keep computing ...

Exhaustive check of all primitive invariants

Recall: we were working through the list of “all” possible invariance groups:



keep computing ...

Exceptional magic

Tabulate the solutions to all $V \otimes \bar{V} \rightarrow V \otimes \bar{V}$ Diophantine conditions

m	8	9	10	12	15	18	24	36
F_4			0	0	3	8	21	52
E_6		0	0	2	8	16	35	78
E_7	0	1	3	9	21	35	66	133
E_8	3	8	14	28	52	78	133	248

Surprise!: all of them are the one and the same Diphantine condition

$$N = \frac{(\ell - 6)(m - 6)}{3} - 72 + \frac{360}{\ell} + \frac{360}{m}$$

!

magically arranging all exceptional families into a

Magic Triangle

Magic Triangle

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
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3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Magic triangle: All solutions of the Diophantine conditions

A brief history of exceptional magic

1975-77: Primitive invariants construction of all semi-simple Lie algebras[1,2], except for the E_8 family.

1979: E_8 family.

1981: Magic Triangle[3]. The total number of citations in the next 22 years: 2 (two).

1996: Deligne[4] conjectures for A_1, A_2, G_2, F_4, E_6, E_7 and E_8 family.

2001: Landsberg and Manivel[5] interpret the Magic Triangle, derive an infinity of higher-dimensional rep formulas.

2002: Deligne and Gross[6]: the Magic Triangle.

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[1] P. Cvitanović, *Phys. Rev.* **D14**, 1536 (1976)

[2] P. Cvitanović, Oxford preprint 40/77 (June 1977); www.nbi.dk/ChaosBook

[3] P. Cvitanović, *Nucl. Phys.* **B188**, 373 (1981)

[4] P. Deligne, *C.R. Acad. Sci. Paris, Sér. I,* **322**, 321 (1996)

[5] J. M. Landsberg and L. Manivel, *Advances in Mathematics* **171**, 59-85 (2002); arXiv:math.AG/0107032, 2001

[6] P. Deligne and B. H. Gross, *C.R. Acad. Sci. Paris, Sér. I,* **335**, 2002 (2002)

Epilogue

“**Why did you do this?**” you might well ask.
OK, here is an answer.

> If gauge invariance of QED and QCD guarantees that all UV and IR divergences cancel, why not also the finite parts?

Electron magnetic moment: each Feynman diagram is of order of 10 to 100, but for gauge invariant subsets a rather surprising thing happens[1]; every subset computed so far adds up to approximately

$$\pm\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^n .$$

If this continues to higher orders, the “zeroth” order approximation to the electron magnetic moment is given by

$$\frac{1}{2}(g-2) = \frac{1}{2}\frac{\alpha}{\pi}\frac{1}{\left(1-\left(\frac{\alpha}{\pi}\right)^2\right)^2} + \text{“corrections”} .$$

[1] P. Cvitanović, “Asymptotic estimates and gauge invariance,” *Nucl. Phys.* **B127**, 176 (1977)

A great heresy

Dyson has shown that the perturbation expansion is an asymptotic series, in the sense that the nth order contribution should be exploding combinatorially

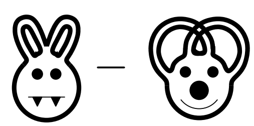
$$\frac{1}{2}(g-2) \approx \cdots + n^n \left(\frac{\alpha}{\pi}\right)^n + \cdots ,$$

and not growing slowly like my estimate

$$\frac{1}{2}(g-2) \approx \cdots + n \left(\frac{\alpha}{\pi}\right)^n + \cdots .$$

I am looking for a simpler gauge theory in which I can compute many orders in perturbation theory and check the conjecture - hence devised fast methods to compute the group weights of many Feynman diagrams in non-Abelian gauge theories.

QCD quarks are supposed to come in three colors. This requires evaluation of SU(3) group theoretic factors, something anyone can do. In the spirit of Teutonic completeness, I wanted to check all possible cases; what would happen if the nucleon consisted of 4 quarks, doodling

 $= n(n^2-1)$,

and so on, and so forth. In no time, and totally unexpectedly, all exceptional Lie groups arose, not from conditions on Cartan lattices, but on the same geometrical footing as the classical invariance groups of quadratic norms, $SO(n)$, $SU(n)$ and $Sp\,(n)$.

No dice. To this day I still do not know how to prove or disprove the conjecture.

Magic ahead

Nobody, but truly nobody in those days showed a glimmer of interest in the exceptional Lie algebra parts of this work, so there was no pressure to publish it before completing it:

1) find the algorithms that reduce any bubble diagram to a number for any semi-simple Lie algebra. The task is accomplished for G_2, but for F_4, E_6, E_7 and E_8 this is still an open problem.

This, perhaps, is only matter of algebra (all of my computations were done by hand, mostly on trains and in airports), but the truly frustrating unanswered question is:

2) Where does the Magic Triangle come from?
3) Why is it symmetric across the diagonal?
4) Is there a mother of all Lie algebras, some complex function which yields the Magic Triangle for a set of integer values?

And then there is a practical issue of unorthodox notation: transferring birdtracks from hand drawings to LaTeX took another 21 years. In this I was rescued by Anders Johansen who undertook drawing some 4,000 birdtracks needed to complete this manuscript, of elegance far outstripping that of the old masters.