Pfettisheim, le 17/01/08.

Bergdolt Georges
e-mail g.bergdolt@orange.fr
Retired from the Centre National de la Recherche Scientifique

I read he paper 'Group Theory' [1] with much interest discovering an unusual appoach and new features.

My objections are about the properties of projectors defined by Young tableaux. The properties of projectors given in section 9.4.2 are valid for the symmetric group $\mathbf{S}_k$ with $k \leq 4$. For $\mathbf{S}_5$ and partition $[3, 2]$ one finds nonorthogonal projectors as shown below. This was already pointed out by Littlewood [2] in his book edited some 50 years ago.

I am not a specialist of birdtracks but I could prove my assertion very easily using that technique. I am rather computer oriented and some 10 years ago I wrote a library of subroutines handling features of the permutation group. The second objection concerns the normalisation constant to make projectors idempotent given in section 9.4.2 of the paper. The value given in the paper differs from the correct value by a factor representing the order of the row and column subgroups.

1 Partition $[3, 2]$.

Projectors for the permutation group $\mathbf{S}_k$ as defined in section 9.4.1 are orthogonal up to $k = 4$. For $\mathbf{S}_5$ the partition $[3, 2]$ admits 5 Young tableaus shown in Fig 1. Let a label Young tableau Y_a, the symmetrizer S_a is the sum of all permutations permuting symbols in rows of Y_a. The antisymmetrizer A_a is the weighted sum of all permutations permuting the symbols in columns of Y_a, with weight ± 1 for even/odd permutations. The unnormalized projector P_a is defined by $P_a = S_a A_a$. The product of the projectors is

$$P_a P_b = S_a A_a S_b A_b$$

In the graphical method advocated by Cvitanovic [1] antisymmetrization is denoted by a black rectangle, symmetrization by a white rectangle, a check for the orthogonality condition is then: two lines coming from a black rectangle enter into the same white rectangle. Applied to the $[3, 2]$ partition of $\mathbf{S}_5$ it turns out that $P_5 P_1 \neq 0$. The birdtrack for the $A_1 S_5$ product (Fig 2) shows that $A_1 S_5$ does not satisfy the orthogonality condition, whereas Fig 3 shows that the birdtrack for the $A_5 S_1$ product satisfies it.

2 Computation of $P_1 P_5$.

Let $\sigma \in A$ be a permutation belonging to the subgroup of column permutations A We have $\mathbf{A}_a \sigma = \pi(\sigma) \mathbf{A}_a$ where $\pi(\sigma)$ is the parity of σ

Let $\sigma \in S$ be a permutation belonging to the subgroup of row permutations S We have $\sigma \mathbf{S}_a = \mathbf{S}_a$

The permutation σ_{15} transforming Y_1 into Y_5 can be factored into transpositions :

$$\sigma_{15} = \tau(2,5)\tau(5,3)\tau(4,2)$$

Here $\tau(2,5)$ belongs to A_1, so $A_1 = -A_1\tau(2,5)$ and $\tau(5,3)\tau(4,2)$ belongs to S_5 hence $\tau(5,3)\tau(4,2)S_5 = S_5$ it follows

$$\begin{aligned} A_1 S_5 &= -A_1\tau(2,5)\tau(5,3)\tau(4,2)S_5 \\ &= -A_1\sigma_{15}P_5 \\ &= -A_1 S_1 \sigma_{15} \\ &= -\sigma_{15} A_5 S_5 \end{aligned}$$

Let σ_{ab} permute the symbols of tableau Y_a to the symbols of tableau Y_b we have [2)]:

$$A_a\sigma_{ab} = \sigma_{ab}A_b, \quad \sigma_{ab}S_b = S_a\sigma_{ab}$$

Hence

$$P_1P_5 = -\sigma_{15}P_5^2 = -P_1^2\sigma_{15}$$

or

$$P_1P_5 = -|Y|\sigma_{15}P_5 = -|Y|P_1\sigma_{15}$$

where $|Y|$ is the number given by the hook rule aplied to the frame Y. The set of projectors P_a $a = 1, \ldots \Delta_Y$ is not closed under multiplication as σ_{15} is not part of the set.

3 Basis of the group algebra.

A set of elements spanning the group algebra of $\mathbf{S}_k$ can be obtained if, following Littlewood[2)] , a set of generalised 2-index projectors is defined by:

$$P_{ab} = S_a\sigma_{ab}A_b$$

The 2-index projectors satisfy :

$$P_{ab}P_{cd} = g_{bc}P_{ad}$$

where $g_{bc} = \pm|Y|$ is a scalar. The only dependence on Young tableau indices b, c is the $\pm$ sign. If an ordering of the symbols in the permutation group is chosen, Young tableaus can be ordered by the first differing symbol, reading the symbols row wise. It can be shown [2)] that $A_aS_b = 0$ if $b < a$, so that g_{ab} is upper triangular.

Let Y, Z label Young frames i.e. partitions, projectors corresponding to different partitions are orthogonal[2)]. The products of the projectors are given by:

$$P_{Yab}P_{Zcd} = \delta_{YZ}g_{Ybc}P_{Yad}$$

A well known result is

$$\sum_{Yab} 1 = \sum_{Y} \Delta_Y^2 = k!$$

where Y is summed over the partitions; a,b over the corresponding Young tableaus. The set of projectors P_{Yab} spans the group algebra of $\mathbf{S}_k$.

4 Normalization constant.

Let Y label partitions i.e. Young frames and $a = 1, \ldots \Delta_Y$ Young tableaux. The normalization factor α_Y for the unnormalized projectors P_a is defined by $(\alpha_Y P_a)^2 = \alpha_Y P_a$. We have

$$\alpha_Y = \frac{\Delta_Y}{k!} = \frac{1}{|Y|}$$

where Δ_Y is the number of Young tableaux, $|Y|$ the number obtained by the hook rule applied to the Young frame Y. The result is given by Littlewood [2)]. The normalization constant given in section 9.4.2 of the paper [1)] differs by a factor $|S_a||A_a|$ where $|S_a|$ is the order of the subgroup of row permutations, $|A_a|$ the order ofthe subgroup of column permutations. It is true that $S_a^2 = |S_a|S_a$ and $A_a^2 = |A_a|A_a$ but S_a and A_a do not commute. The value of α_Y has been checked by computer algebra by a direct method; the unnormalised projector is expanded in permutations. It can be shown that the coefficients are restricted to $-1, 0, +1$. The product is then computed using the product of permutations. The result is in agreement with $P_a^2 = |Y|P_a$.

References:

1. P.Cvitanovic, Group Theory, PU copyeditor version 8.7 September 27, 2007.

2. D.E.Littlewood, The Theory of Characters and Matrix Representations of Groups, Oxford Clarendon Press 1958.

3. G.Bergdolt, An algebraic approach to representations of the permutation group, arXiv:math/0112117v1.

Fig. 1 Young tableaux

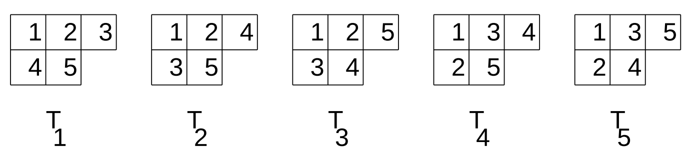

Fig. 2

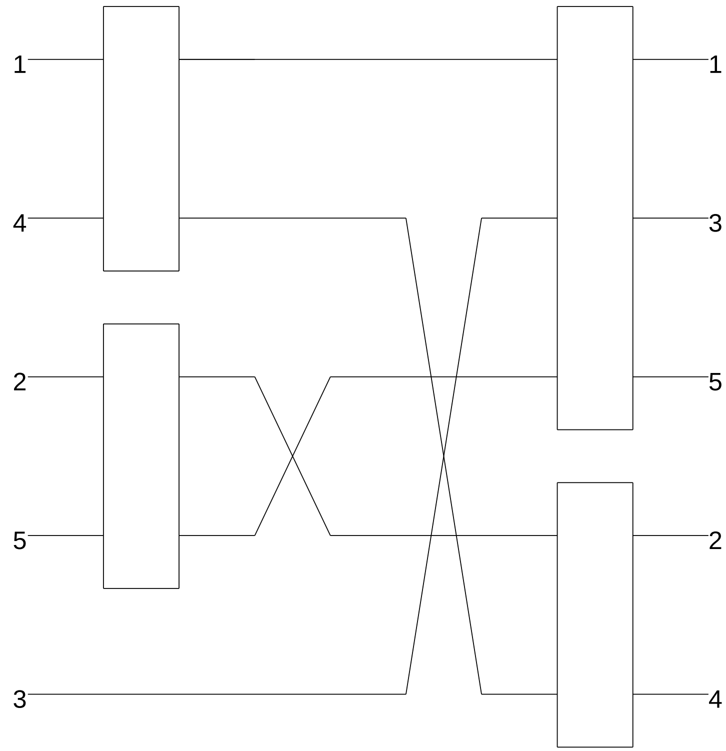

Fig.3